Application of the Arnoldi Method in FEM Analysis of Waveguides

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***Abstract*—This letter presents the application of the Arnoldi method to the solution of generalized nonsymmetric sparse eigenproblems which arise in the waveguide analysis involving the finite element method. To assess the efficiency of the Arnoldi method, the solution time is compared against the time required by the subspace iteration algorithm. It is found that the Arnoldi method converges much faster and gives significant CPU time savings.**

***Index Terms*— Arnoldi method, finite-element methods, nonsymmetric sparse eigenproblems.**

I. Introduction

A FINITE-ELEMENT method (FEM) is one of the most versatile techniques of solving partial differential equations and many authors [1]–[5] advocate the application of this method to the analysis of arbitrarily shaped inhomogeneously loaded waveguides. If a guide is strictly bidirectional [6] the finite element analysis leads to the generalized eigenvalue problem [5]

$$\underline{\underline{A}}\,\underline{x} = \gamma^2 \underline{\underline{B}}\,\underline{x} \tag{1}$$

where γ^2 is an eigenvalue and $\underline{x}$ denotes a vector of field expansion coefficients. Matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are, in general, sparse and nonsymmetric.

The choice of a method used to solve the above problem determines the overall computation time and is one of the crucial steps in the FEM analysis. Because of the nonsymmetry of the matrices and the nondiagonal structure of matrix $\underline{\underline{B}}$ the choice of numerical methods for solution of (1) taking into account the sparsity is limited. To the authors' knowledge the most efficient technique used so far in the FEM analysis of waveguides is the sparse version of the subspace iteration (SI) technique [3]–[5]. Recent studies showed that, depending on the application, the performance of the software based on SI may be superior or inferior [7] to the performance of the software based on the Krylov space concept such as the Arnoldi method [8]. For this reason, in this letter, we compare the efficiency of the SI with the Arnoldi method in the waveguide analysis. It has to be noted that the Arnoldi method was shown to give very good results in solution standard dense [9] and sparse [10] eigenproblems obtained by means of the method of moments or finite-difference frequency-domain method (FDFD), respectively, but has not yet been tested in the generalized eigenproblem which arise in the FEM analysis of waveguides.

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II. Solution of Sparse Nonsymmetric Eigenproblem

Numerical techniques for computing eigenvalues of sparse nonsymmetric problems include the subspace iteration method and the Krylov space methods [8]. The subspace iteration method starts with the initial $n \times m$ matrix (where $m \ll n$ and n is the size of the sparse matrix to be analyzed) and generates a sequence of matrices which converge to the invariant subspace of a sparse matrix corresponding to the m dominant eigenvalues. An example of the Krylov subspace method is the Arnoldi algorithm. The Arnoldi method starting with a trial vector $\underline{x}$ builds an orthogonal basis in the Krylov subspace. For many years the Arnoldi method was considered less efficient than SI because of the higher memory requirements. However, this difficulty has been overcome in the recent years with the introduction of explicitly [8] or implicitly [11] restarted iteration.

In solving large sparse problems one is usually interested in finding only selected eigenvalues which may be located in various parts of the spectrum. For instance, in waveguide problems one is typically interested in a few dominant modes which correspond to the eigenvalues with the largest real part but the smallest magnitude [5]. The most suitable technique for finding the dominant modes involves the shift-invert strategy in which eigenproblem (1) is converted to the eigenproblem

$$(\underline{\underline{A}} - \sigma\underline{\underline{B}})^{-1}\underline{\underline{B}}\,\underline{x} = \frac{1}{\gamma^2 - \sigma}\underline{x} \tag{2}$$

where σ is the shift. When an iterative solver is applied, the product of matrix operator and some varying vector $\underline{x}$ is repeatedly calculated. In the modified eigenproblem (2), the matrix operator is the product of the inverse of matrix $(\underline{\underline{A}} - \sigma\underline{\underline{B}})$ and matrix $\underline{\underline{B}}$. Instead of calculating the inverse directly, a sparse LU decomposition of the matrix is performed. Consequently, when $\underline{y} = (\underline{\underline{A}} - \sigma\underline{\underline{B}})^{-1}\underline{\underline{B}}\,\underline{x}$ product is required, a linear system of equations $(\underline{\underline{A}} - \sigma\underline{\underline{B}})\underline{y} = \underline{\underline{B}}\underline{x}$ is solved instead.

The convergence rate in the shift-invert mode in iterative methods depends on the shift σ. In the waveguide analysis it is convenient to choose the shift so that $\sigma > k_0^2\epsilon_{\max}$ where k_0 is the wavenumber for a plain wave in vacuum. In that case the dominant modes correspond to the eigenvalues of (2) possessing the largest magnitude.

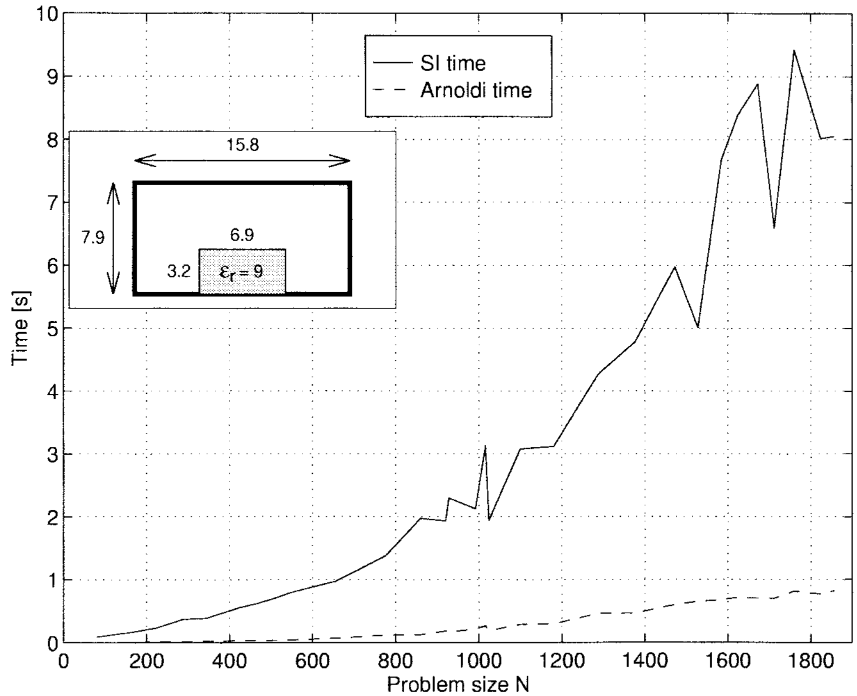

Fig. 1. Analyzed structure (dimensions are in millimeters) and calculation times (CPU time SGI power challenge) of the subspace iteration and the Arnoldi methods for odd modes versus the problem size. The modes were calculated at 12 GHz.

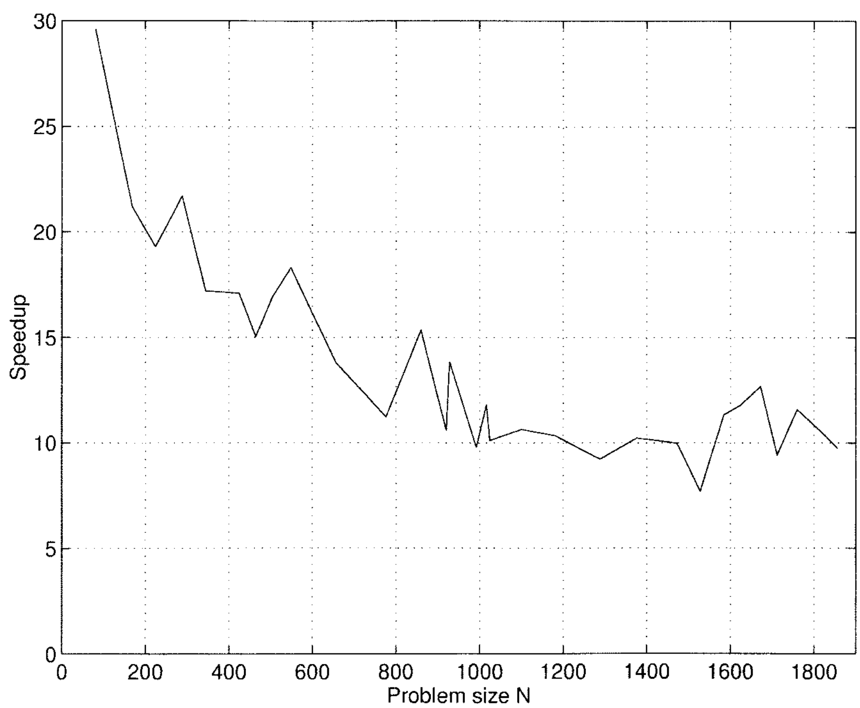

Fig. 2. Speedup of the Arnoldi method over the subspace iteration.

III. Results

To compare the efficiency of the Arnoldi method versus the subspace iteration we used the FEM code developed at UCL London [3] to compute the four dominant modes of a symmetric structure of the image guide shown in Fig. 1. The code, which was originally designed to work with the SI, was modified so that different solvers could be used. Using the modified code we compared the SI based on the algorithm described in [5] with the implicitly restarted Arnoldi (IRA) method available in the form of a public domain library ARPACK.[1]

[1] http://www.caam.rice.edu/software/ARPACK.

The shift-invert approach was applied in the form of (2). Before iteration, the sparse LU decomposition was performed on the matrix $(\underline{\underline{A}} - \sigma \underline{\underline{B}})$. In order to ensure convergence of the SI method we had to use different values of σ depending on calculated mode type, $\sigma = 16k_0^2$ for odd and $\sigma = 12k_0^2$ for even modes. In the IRA method the shifts were selected in the same manner. The size of subspace in each of the methods was fixed at $m = 8$. Relative accuracy of the calculated eigenvalues was 10^{-6} in both methods.

Calculation times of four dominant odd modes for SI and IRA methods are presented in Fig. 1. The Arnoldi method is seen to be significantly faster than the subspace iteration. Speedup of the Arnoldi method over SI, presented in Fig. 2, is greater than 8 for almost all problem sizes. Very similar observations were made for calculation of four dominant even modes. A better performance of the IRA method comes from two factors. One reason is a faster convergence rate. In our tests the IRA method converged in nine iterations while SI required ≈30 iterations before the convergence criteria were satisfied. Additionally the IRA iterations involved fewer (costly) solutions of a linear system $(\underline{\underline{A}} - \sigma \underline{\underline{B}})\underline{y} = \underline{\underline{B}}\,\underline{x}$. The IRA method required the system to be solved <30 times while the 30 iterations in the SI method was associated with 240 solutions. Comparison of the number of solution steps gives the figure of 8, which is in agreement with the data in Fig. 2.

IV. Conclusions

Tests performed show that the Arnoldi method with implicit restart is more efficient than the subspace iteration method for solving nonsymmetrical sparse eigenproblems arising in the FEM analysis of dielectric waveguides.

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